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Kink production in the presence of random distributed impurities

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Production of kinks during a quench in an overdamped regime of ϕ^4 model is investigated. An influence of random distributed impurities on defect production is studied.

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I. INTRODUCTION

In recent years there has been growing interest in extracting information about production of topological defects in classical field theories at finite temperatures. The motivation for these investigations is provided by the possibility that the topological defects produced during the transitions at GUT scale can be responsible for providing inhomogeneities sufficient for galaxy formation in the early Universe [1]. These investigations are encouraged by the easy access to condensed matter systems that provide the field for experimental verification of the possible theoretical scenarios [2]. Particulary interesting in this context is the behavior of the quantum liquids [3].

The dynamics of the transition was first explained by Kibble and Zurek. They proposed two complementary scenarios. Kibble in his scenario [4] emphasized an importance of the nucleation processes in creation of the topological defects during the first order phase transition, on the other hand, Zurek stressed that the nonequilibrium dynamics is crucial for behavior of the system during the second order phase transition [5].

Zurek noticed that, as a result of critical slowing down, the state of the system that crosses the critical region at a finite pace will inevitably cease to keep up with the changes of thermodynamic parameters at some point sufficiently near the critical temperature. In a homogenous quench, this will happen everywhere at the same time. The time when the system loses the ability to respond for the changes of external parameters is called freeze-in time. The correlation length at the same time after the transition, i.e., at instant of time when the system regains capacity to respond to changes of external parameters (freeze-out time), determines the initial density of defect network. After this discovery the progress in this matter concerned mainly on the description of the systems driven by white Gaussian noise [5,6].

On the other hand, we know that, in most of the physical contexts, the population of the condensed matter systems by impurities is inevitable outcome of their preparation. In spite of the fact that the presence of impurities and admixtures may completely change properties of the system, prevailed part of the obtained hitherto results concern homogenous medium.

The phase transition in liquid crystals is a nice example of the system that can be easily studied experimentally and also can be occupied by impurities. Moreover, liquid crystals allow us to study the production of topological defects. However, the transition in this system is a little different from the pure second order because near critical temperature a nonzero energy barrier occurs. In these settings the pure false vacuum instability that is described by Zurek scenario is camouflaged by the thermal nucleation.

The other systems that can be easily contaminated and are described by Zurek scenario are superconductors of second type. They seem to be particularly useful in testing an influence of impurities and admixtures on defect formation.

The other context is provided by superfluid liquids. Unfortunately the solubility of all foreign materials in liquid helium is near zero. The only way of contamination of Helium 3 or 4 is by using an aerogel technique [7]. In this context the question rise if the level of contamination provided by the aerogel, which is a matrix of randomly arranged silica filaments of nanometer diameter, can have any observable effect on defect production?

The aim of this paper is to discuss an influence of the randomly distributed spatial inhomogeneities (which represent impurities in the system) on defect production during a quench. The model to be studied here is a kink-bearing ϕ^4 field theory in $(1+1)$ dimensions. This model is so popular because its properties are representative for many physical systems that undergo a transition from a spatially uniform to one of lower symmetry states.

The paper is organized as follows. The following section contains the generalization of the Halperin formula to description of the system populated by randomly distributed impurities. In Sec. III we consider an example of spatially correlated noise of Ornstein-Uhlenbeck type. Section IV contains remarks.

II. THE HALPERIN FORMULA FOR RANDOMLY DISTRIBUTED IMPURITIES

The number density of zeros of the scalar field is calculated as a sum over all points x_i, defined by the equation $\phi(t,x_i)=0$ [8]

$$n\equiv\{n(t,x)\}=\left\{\lim_{L\to 0}\frac{\langle N\rangle}{2L}\right\}=\left\{\lim_{L\to 0}\frac{1}{2L}\left\langle\sum_i\frac{|\phi'(t,x_i)|}{|\phi'(t,x_i)|}\right\rangle\right\}. \tag{1}$$

The brackets in this formula denote averaging with respect to realizations of the temperature noise $\langle\cdots\rangle$, and the spatial distributions of the impurities $\{\cdots\}$. We calculate the number of produced defects in the overdamped ϕ^4 model

$$\hat{\gamma}\partial_t\phi(t,x)=\partial_x^2\phi(t,x)-a(t)\phi(t,x)-\lambda\phi^3(t,x)+\eta(t,x)+\mathcal{D}(t,x), \tag{2}$$

where $\eta(t,x)$ represents a temperature noise (it is assumed that this is a purely white Gaussian noise), $\mathcal{D}(t,x)$ represents a force coming from randomly distributed impurities and $\hat{\gamma}$ is an integral operator [9] that follows from a linear fluctuation-dissipation relation (in case of white Gaussian noise it is just damping constant γ). We assume that random forces are defined by the cumulants,

$$\langle\eta(t,x)\rangle=0,$$

$$\langle\eta(t,x)\eta(t',x')\rangle=\frac{2\pi\gamma}{\beta}\delta(x-x')\delta(t-t'), \tag{3}$$

and

$$\{\mathcal{D}(t,x)\}=0,$$

$$\{\mathcal{D}(t,x)\mathcal{D}(t',x')\}=\frac{1}{\beta}W(|x-x'|)\delta(t-t'). \tag{4}$$

We intend to calculate the number of kinks produced during a quench just after the transition time, i.e., at freeze-out time. We know that just before and after instant of transition the order parameter is so small that a cubic term is negligible in comparison with linear terms. We also assume that in linear approximation an influence of the thermal noise and the random impurity force on the evolution of the order parameter can be described by additive variables $\phi(t,x)=\psi(t,x)+u(t,x)$ that satisfy equations of motion,

$$\gamma\partial_t\psi(t,x)=\partial_x^2\psi(t,x)-a(t)\psi(t,x)+\eta(t,x), \tag{5}$$

$$\hat{\gamma}_*\partial_t u(t,x)=\partial_x^2 u(t,x)-a(t)u(t,x)+\mathcal{D}(t,x). \tag{6}$$

In this situation the number density of produced defects is described by the formula

$$n=\frac{1}{\pi}\sqrt{\frac{\langle\psi'^2\rangle}{\langle\psi^2\rangle}}\left\{\exp\left[-\frac{u^2}{2\langle\psi^2\rangle}-\frac{u'^2}{2\langle\psi'^2\rangle}\right]\right\}+\frac{1}{\pi}\frac{1}{\sqrt{\langle\psi^2\rangle\langle\psi'^2\rangle}}\left\{u'e^{-u^2/2\langle\psi^2\rangle}\int_0^{u'}d\tilde{u}'e^{-\tilde{u}'^2/2\langle\psi'^2\rangle}\right\}, \tag{7}$$

where we have to calculate averages over realizations of the impurity force. In this purpose first we introduce a new integration variable $s=\tilde{u}'/u'$ and then we replace average of the products by the product of averages, i.e.,

$$\left\{\exp\left[-\frac{u^2}{2\langle\psi^2\rangle}-\frac{u'^2}{2\langle\psi'^2\rangle}\right]\right\}=\{e^{-u^2/2\langle\psi^2\rangle}\}\{e^{-u'^2/2\langle\psi'^2\rangle}\}, \tag{8}$$

$$\left\{u'^2e^{-u^2/2\langle\psi^2\rangle}\int_0^1 ds\exp\left(-\frac{s^2u'^2}{2\langle\psi'^2\rangle}\right)\right\}=\{e^{-u^2/2\langle\psi^2\rangle}\}\left\{u'^2\int_0^1 ds\exp\left(-\frac{s^2u'^2}{2\langle\psi'^2\rangle}\right)\right\}. \tag{9}$$

This replacements are possible because for cumulants (8) and (9) odd terms in Wick expansion of the exponents disappear (see [8] for details). Now we have to calculate the averages of the exponents. For instance,

$$\{e^{-u^2/2\langle\psi^2\rangle}\}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\left\{\left(\frac{u^2}{\langle\psi^2\rangle}\right)^n\right\}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\left(\frac{\{u^2\}}{\langle\psi^2\rangle}\right)^n, \tag{10}$$

where we used a Wick theorem. The last series can be easily summed up

$$\{e^{-u^2/2\langle\psi^2\rangle}\}=\frac{1}{\sqrt{1+\frac{\{u^2\}}{\langle\psi^2\rangle}}}. \tag{11}$$

In similar way we can sum up the other exponents. An integral we average using the same type of expansion and then by changing an order of a sum and the integral we obtain

$$\int_0^1 ds\left\{u'^2\exp\left(-\frac{s^2u'^2}{2\langle\psi'^2\rangle}\right)\right\}=\frac{\{u'^2\}}{\sqrt{1+\frac{\{u'^2\}}{\langle\psi'^2\rangle}}}. \tag{12}$$

Now we have averaged number density of zeros produced in the presence of random impurity force,

$$n=\frac{1}{\pi}\sqrt{\frac{\langle\psi'^2\rangle}{\langle\psi^2\rangle}}\frac{1}{\sqrt{1+\frac{\{u^2\}}{\langle\psi^2\rangle}}}\frac{1}{\sqrt{1+\frac{\{u'^2\}}{\langle\psi'^2\rangle}}}+\frac{1}{\pi}\frac{1}{\sqrt{\langle\psi^2\rangle\langle\psi'^2\rangle}}\frac{1}{\sqrt{1+\frac{\{u^2\}}{\langle\psi^2\rangle}}}\frac{\{u'^2\}}{\sqrt{1+\frac{\{u'^2\}}{\langle\psi'^2\rangle}}}. \tag{13}$$

Simple algebra provides the formula that in elegant way corresponds to Liu-Mazenko-Halperin formula

$$n=\frac{1}{\pi}\sqrt{\frac{\langle\psi'^2\rangle+\{u'^2\}}{\langle\psi^2\rangle+\{u^2\}}}. \tag{14}$$

Let us notice that this formula provides the way of estimation of number density of defects in any spatially correlated noise. At this point it is worth to stress that there exists also the other possible way of entering the noises characterized by a new length scale. It seems that the presence of quasiparticles in superfluid liquids provides a bath equipped in new length and time scales such as the mean free path that can make the noise spatially correlated or even colorful [10]. The Liu-Mazenko-Halperin [11] formula is restored in case when the random impurity force is absent in Eq (2), i.e., $\mathcal{D}(t,x)=0$ and, therefore, $\{u^2\}=\{u'^2\}=0$

$$n=\frac{1}{\pi}\sqrt{\frac{\langle\psi'^2\rangle}{\langle\psi^2\rangle}}. \tag{15}$$

III. AN INFLUENCE OF RANDOM SPATIALLY CORRELATED IMPURITY FORCE ON DEFECT PRODUCTION

The Fourier transformation of the equations of motion $\psi(t,x)=\int_{-\infty}^{\infty}dk e^{ikx}\tilde{\psi}(t,k)$ enables to find Green functions of the equations (5) and (6)

$$\gamma\partial_t\tilde{\psi}(t,k)+k^2\tilde{\psi}(t,k)+a(t)\tilde{\psi}(t,k)=\tilde{\eta}(t,k), \tag{16}$$

$$\tilde{W}(k)\partial_t\tilde{u}(t,k)+k^2\tilde{u}(t,k)+a(t)\tilde{u}(t,k)=\tilde{\mathcal{D}}(t,k). \tag{17}$$

Actually in further calculus we choose units in such a way as to have $\gamma=1$. For instance, the solution of the Eq. (16) have the form

$$\tilde{\psi}(t,k)=\int_{-\infty}^{t}dt_1\exp\left\{-\int_{t_1}^{t}dt_2[k^2+a(t_2)]\right\}\tilde{\eta}(t_1,k). \tag{18}$$

The power spectrum for the variable ψ is defined by the equality

$$\langle\tilde{\psi}(t,k)\tilde{\psi}(t',k')\rangle_{t=t'}=\mathcal{P}_{\psi}(t,k)\,\delta(k-k'). \tag{19}$$

The solution (18) together with the thermal noise cumulants allows to find the power spectrum. For sufficiently late time it has the form

$$\mathcal{P}_{\psi}(t,k)\approx\frac{1}{\beta}\sqrt{\pi\tau}e^{t^2/\tau}e^{-2k^2t}e^{\tau k^4}. \tag{20}$$

This power spectrum blows up at $t_e\approx\sqrt{\tau}$. This is the time when the linear approximation fails, therefore, it is identified with the freeze-out time (in fact, this is the instant of time when the nonlinear dynamics enters into description of the system). On the other hand power spectrum of the impurity component is defined by the equal time correlator of the second function

$$\{\tilde{u}(t,k)\tilde{u}(t',k')\}_{t=t'}=\mathcal{P}_u(t,k)\,\delta(k-k'). \tag{21}$$

The power spectrum in this case is modified by the function $\tilde{W}$ that defines the noise cummulant

$$\{\tilde{\mathcal{D}}(t,k)\tilde{\mathcal{D}}(t',k')\}=\frac{1}{\beta}\tilde{W}(|k|)\,\delta(k-k')\,\delta(t-t'). \tag{22}$$

In the above mentioned regime the power spectrum has the form

$$\mathcal{P}_u(t,k)\approx\frac{1}{\beta\tilde{W}}\sqrt{\pi\tau}e^{t^2/\tilde{W}\tau}e^{-2k^2t/\tilde{W}}e^{\tau k^4/\tilde{W}}. \tag{23}$$

The modes connected with the impurity component blows up at the time $t_e\approx\sqrt{\tilde{W}(k)}\,\tau$. In fact, different modes blow up at different times. The number of produced zeros is determined by the obtained in preceding section formula (14). This formula can also be rewritten with the use of the power spectrums of the thermal and impurity components

$$n=\frac{1}{\pi}\sqrt{\frac{\int_0^{k_m}dk k^2\mathcal{P}_{\psi}+\int_0^{k'_m}dk k^2\mathcal{P}_u}{\int_0^{k_m}dk\mathcal{P}_{\psi}+\int_0^{k'_m}dk\mathcal{P}_u}}. \tag{24}$$

An obvious fact is that a number of produced zeros is infinite. This is a consequence of the structure of the noise. Although zeros are produced on all scales not all zeros can produce kink structures. We know that only zeros separated at least by the correlation length can produce stable kinks. Stable and unstable modes can be determined from linearized equations of motion (5) and (6). In the language of the Fourier components the stable long range spatial structures can be produced by the unstable modes of the Eqs. (16) and (17) because only those modes can grow to produce stable kink structures. Therefore the integration in the formula (24) have to be restricted to momentums smaller than cutoff determined from the equation $k_m^2-t_e/\tau=0$. Thus the cutoff in momentum for thermal component is the following $k_m=\sqrt{t_e/\tau}=\tau^{-1/4}$. In case of the impurity component the cutoff strongly depends on the model. This cutoff is a solution of the equation $k_m'^4=\tilde{W}(|k'_m|)/\tau$. We assume a noise correlator amplitude in the form $\tilde{W}(|k|)=a/(1+L^2k^2)$. Under choice $a=(1/\pi\mathcal{A}L)$ this amplitude is the Fourier transformation of the well known Ornstein-Uhlenbeck distribution $W(|x|)=\mathcal{A}e^{-|x|/L}$ that in the limit $L\to 0$ reduces $\mathcal{D}(t,x)$ to white Gaussian noise. We consider two cases. (1) $L^4\gg\tau$ then $k'_m\approx 2/3\tau^{1/4}$, (2) $L^4\sim\tau$ then $k'_m\approx 1/L$, where k'_m was calculated from equation $k_m'^4=\tilde{W}(|k'_m|)/\tau$. The number density of produced defects in the first regime can be approximated as follows:

$$n\approx 0.1\frac{1}{(L\tau^{1/4})^{1/2}}. \tag{25}$$

We see that the characteristic length scale of the impurity noise enters the number density of the produced kinks. In the second regime the number density formula shows the continuous transition from thermal noise dominance to impurity dominated regime [9]

$$n \approx \frac{1}{\pi}\sqrt{\frac{0.43b/\tau^{1/2}+0.34c/L^2}{1.81b+0.83c}}, \qquad (26)$$

where $b=1/\tau^{1/4}$ and $c=L/\sqrt{\tau}$. We see that if the impurities are absent or week (i.e., $\mathcal{A}$ is small) we have classical scaling $n \sim 1/\tau^{1/4}$ [5]. On the other hand, if transition takes place for very low temperatures then thermal fluctuations are almost absent and the number density is determined by the impurity noise length scale $n \sim 1/L$.

IV. REMARKS

In present paper we propose an extension of the description of the production of the topological defects to the systems populated by the impurities. In present approach, we demonstrate the comprehensive treatment of impurities as a spatially correlated additive noise. This approach is a natural development of the description of the impurities and admixtures proposed in paper [8]. We know that there exists two equally important components of the dynamics leading to production of the topological defects. First is a noise that provides the large number of zeros, which are candidates for future kinks. The second component is a correlation length of the order parameter that provides a length scale describing minimal separation of kinks. The obvious fact is that if zeros are much closer to each other than the correlation length, than the correlation length cutoff completely determines the density of produced kinks. This is a case dominated by the white Gaussian thermal noise. On the other hand, if the spatially correlated noise prevails then the density of the initial network of zeros is mainly determined by the noise characteristic length scale. It is reasonable that if at freeze-out time zeros are separated by the distance larger than the correlation length then the density of produced kinks is smaller than the density of kinks predicted in previous case. In the second regime the noise length scale enters the number density of produced kinks formula. We have also shown an example of the spatial Ornstein-Uhlenbeck noise the entrance between two regimes. In the regime dominated by the thermal noise the typical scaling is captured, i.e., $n \sim 1/\tau^{1/4}$. On the other hand, if the impurity noise dominates in the system then the kink distribution is determined by the noise characteristic length scale $n \sim 1/L$.

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